SUMMARY OF CHAPTER 4

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CONCLUSION (CHAPTER 4)

Notation 1 $|S|$ =*number of vectors in* S.

CASE (I) To prove the set $S = \{\underline{v_1}, \underline{v_2}, ..., \underline{v_n}\}$ is span for a vector space V, we have to show that for any vector $u \in V$ we have

$$c_1\underline{v_1} + c_2\underline{v_2} + ... + c_n\underline{v_n} = \underline{u} \quad \text{for scalars } c_i \in \mathbb{R}$$

(1) If the system $Ac = \underline{u}$ is a square system (i.e. $|S| = \dim(V)$), we find the determinant of $|A|$, we have two cases:

(a) $|A| \neq 0$, then the system has a unique solution. So any vector in V can be written as a linear combination of the vectors of S. Thus S spans V.

(b) $|A| = 0$, since $|S| = \dim(V)$ (by theorem 4.12). We prove that S is linearly dependent and hence S does not span V.

OR **"rarely to use"** find the augmented matrix of the system and use Gaussian elimination to deduce if the system has infinite number of solutions (so S spans V) or no solution (so S does not span$V$).

(2) **"rarely occurs"** If the system $A\underline{c} = \underline{u}$ is not square system (i.e. $|S| \neq \dim(V)$), we should find the

augmented matrix and use Gaussian elimination, then consider the three cases of solutions

★No solution (not span)
★Infinite number of solutions (S is a spans V)
★Unique solution (S is a spans V)

CASE (II) To prove the set $S = \{\underline{v_1}, \underline{v_2}, ..., \underline{v_n}\}$ is linearly independent, we have to show that if the vector equation

$$c_1\underline{v_1} + c_2\underline{v_2} + ... + c_n\underline{v_n} = 0 \text{ has only the trivial solution,}$$

(1) If $|S| = 2$, then use (theorem 4.8) $\underline{v_1} \neq k\underline{v_2} \Longleftrightarrow S$ is linearly independent

(2) If $|S| > 2$, see if the homogeneous system $A\underline{c} = 0$ is a square system (i.e. $|S| = \dim(V)$), we find the determinant of $|A|$, we have two cases:

(a) $|A| \neq 0$, then the homogeneous system has a unique solution which is the trivial solution. So S is linearly independent set of vectors.

(b) $|A| = 0$, then there is infinite number of solutions. So S is linearly dependent set of vectors.

(3) If the homogeneous system $A\underline{c} = 0$ is not square system (i.e. $|S| \neq \dim(V)$), we should find the augmented matrix and use Gaussian elimination, then consider the two cases of solutions

★Infinite number of solutions (S is linearly dependent)
★ Unique solution which is the trivial solution (S is linearly independent)

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IMPORTANT THEOREMS

Theorem 2 *(4.8) Two vectors $\underline{u}$ and $\underline{v}$ in a vector space V are linearly dependent iff one vectoer is a scalar multiple of the other. (i.e. $\underline{u}= k\underline{v}$)*

Theorem 3 *(4.10) If $S = \{\underline{v_1}, \underline{v_2}, ..., \underline{v_n}\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependant.*

Theorem 4 *(4.11) If a vector space V has one basis with n vectors, then every basis for V has n vectors.*

Theorem 5 *(4.12) Let V be a vector space of dimension n.*

1. *If $S = \{\underline{v_1}, \underline{v_2}, ..., \underline{v_n}\}$ is linearly independent set of vectors in V, then S is a basis for V.*
2. *If $S = \{\underline{v_1}, \underline{v_2}, ..., \underline{v_n}\}$ spans V, then S is a basis for V.*

CONCLUSION (CHAPTER 4)

Let $\dim(V) = n$, and $S = \{\underline{v_1}, \underline{v_2}, ..., \underline{v_k}\} \subseteq V$. We have three cases concerning k

1. If $k > n$
2. If $k < n$
3. If $k = n$

....

(1) If $k > n$, then

(i) is linearly dependent set (by theorem 4.10)

(ii) S may span V or S may not span V.

(ii) S span V if there exists $S_1 \subset S$ such that $|S_1| = n$ and S_1 is linearly independent

Example 6 $V = R^2$, $S = \left\{ \underbrace{(1,2)}_{\underline{v_1}}, \underbrace{(2,3)}_{\underline{v_2}}, \underbrace{(3,4)}_{\underline{v_3}} \right\}$ *is linearly dependent* ($\underline{v_3} = -\underline{v_1} + 2\underline{v_2}$) *and span* R^2 , *Because there exist*

$$S_1 = \{\underbrace{(1,2)}_{\underline{v_1}}, \underbrace{(2,3)}_{\underline{v_2}}\}$$

such that $S_1 \subset S$ *and* S_1 *is linearly independent* ($\underline{v_1} \neq k\underline{v_2}$)

Example 7 $V = R^2$, $S = \left\{ \underbrace{(1,2)}_{\underline{v_1}}, \underbrace{(2,4)}_{\underline{v_2}}, \underbrace{(3,6)}_{\underline{v_3}} \right\}$ *is linearly dependent* ($\underline{v_3} = \underline{v_1} + \underline{v_2}$) *and* S *cannot span* R^2 , *Because there exist no* S_1 *such that* $S_1 \subset S$ *and* S_1 *is linearly independent* ($\underline{v_1} \neq k\underline{v_2}$)

For example $S_1 = \{\underbrace{(1,2)}_{\underline{v_1}}, \underbrace{(2,4)}_{\underline{v_2}}\}$ *not linearly independent* $\underline{v_1} = 2\underline{v_2}$

For example $S_1 = \{\underbrace{(1,2)}_{\underline{v_1}}, \underbrace{(3,6)}_{\underline{v_3}}\}$ *not linearly independent* $\underline{v_1} = 3\underline{v_3}$

For example $S_1 = \{\underbrace{(2,4)}_{\underline{v_2}}, \underbrace{(3,6)}_{\underline{v_3}}\}$ *not linearly independent* $\underline{v_2} = \frac{3}{2}\underline{v_2}$

(2) If $k < n$, then

(i) S cannot span V (by theorem 4.11)

(ii) S may linearly dependent or may linearly independent.

Example 8 $V = R^3$, $S = \{\underbrace{(1,2,0)}_{\underline{v_1}}, \underbrace{(2,4,0)}_{\underline{v_2}}\}$ *is linearly dependent* $\left(\underline{v_1} = 2\underline{v_2}\right)$

Example 9 $V = R^3$, $S = \{\underbrace{(1,2,0)}_{\underline{v_1}}, \underbrace{(2,6,1)}_{\underline{v_2}}\}$ *is linearly independent* $\left(\underline{v_1} \neq k\underline{v_2}\right)$

....

(3) $k = n$, then by (theorem 4.12)

$$(i) S \text{ is linearly independent} \Rightarrow S \text{ is a basis for } V$$
$$(ii) S \text{ spans } V \Rightarrow S \text{ is a basis for } V$$

i.e.

$$S \text{ is linearly independent} \Leftrightarrow S \text{ span } V$$
$$S \text{ is linearly dependent} \Leftrightarrow S \text{ cannot span } V$$

Example 10 $V = R^2$, $S = \{(4,6), (6,9)\}$ *is linearly dependent* $((4,6) = \frac{2}{3}(6,9)$, *hence* S *cannot span* R^2.

Example 11 $V = R^2$, $S = \{(1,2), (2,3)\}$ *is linearly independent* $((1,2) \neq k(2,3))$, *hence* S *spans* R^2.

Problem 12 *Determine whether S is a basis or not for the given space, use the above summery and try to explain the two conditions of the basis (L.I. & Span)*

1. $V = R^2$, $S = \{(-1, 2), (1, -2), (2, 4)\}$

2. $V = R^2$, $S = \{(3, -2), (4, 5)\}$

3. $V = R^2$, $S = \{(-4, 5), (0, 0)\}$

4. $V = R^2$, $S = \{(6, -5), (12, -10)\}$

5. $V = R^3$, $S = \{(2, 1, -2), (-2, -1, 2), (4, 2, -4)\}$

6. $V = R^3$, $S = \{(7, 0, 3), (8, -4, 1)\}$

7. $V = R^4$, $S = \{(-1, 2, 0, 0), (2, 0, -1, 0), (3, 0, 0, 4), (0, 0, 5, 0)\}$

8. $V = R^2$, $S = \{(5, 0), (5, -4)\}$

9. $V = R^2$, $S = \{(-1, 2), (2, -4)\}$

10. $V = R^2$, $S = \{(-1, 4), (4, -1), (1, 1)\}$

11. $V = R^3$, $S = \{(6, 7, 6), (3, 2, -4), (1, -3, 2)\}$

12. $V = R^3$, $S = \{(-2, 5, 0), (4, 6, 3)\}$

13. $V = R^3$, $S = \{(1, -2, 0), (0, 0, 1), (-1, 2, 0)\}$

14. $V = R^3$, $S = \{(1, -4, 1), (6, 3, 2)\}$

15. $V = R^3$, $S = \{(4, 3, 2), (0, 3, 2), (0, 0, 2)\}$